Marginal Value Theorem

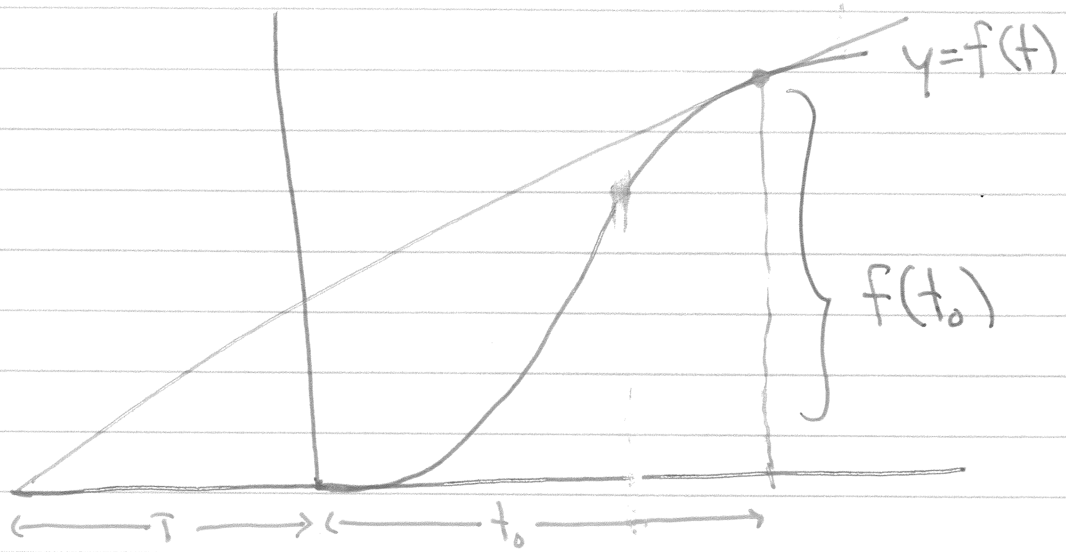
Collection function $f(t)$

Travel time T

$\rightsquigarrow$ Avg rate $R(t) = \dfrac{f(t)}{t+T}$

Then, the optimum t_0 satisfies

$$\boxed{\begin{array}{cc} f'(t_0) = & R(t_0) \\ \text{instantaneous} & \text{Average} \end{array}}$$



Data Fitting

Concentration

Q: Antibiotic

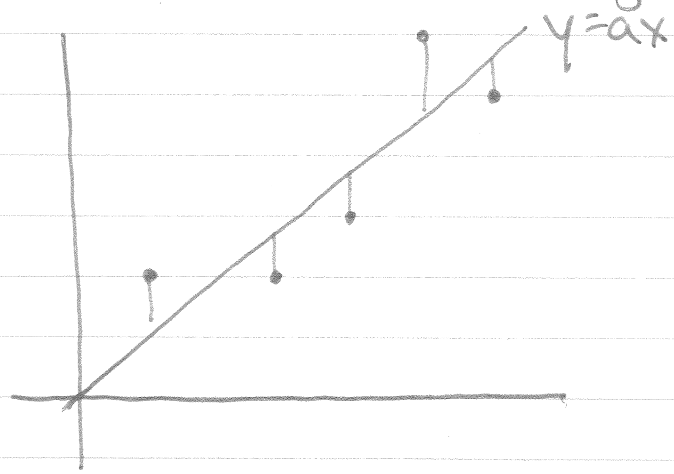
Trial #	% of Bacteria Remaining	Concentration
1	19%	0.1
2	24%	0.2
3	17%	0.1
4	28%	0.2

Trial 5: Expect average

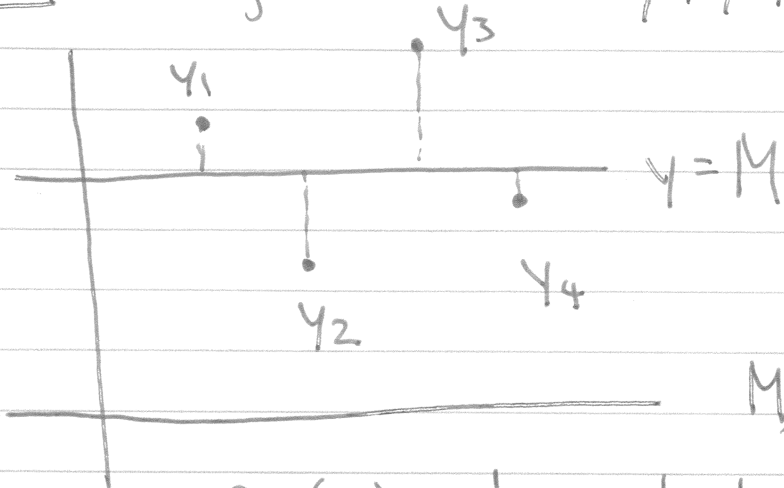
$$\frac{19+24+17+28}{4} = 22\%$$

General Q: Given data $X_1, X_2, \ldots, X_n$, how to create a statistical model which predicts future data?

Approach: Pick a model which minimizes the error ie, deviation from being a perfect fit.



Ex 1: Average of numbers y_1, y_2, y_3, y_4.



Residuals: $(y_1 - M)$, $(y_2 - M)$, $(y_3 - M)$, $(y_4 - M)$.

Minimize residuals.

$$SR(M) = |y_1 - M| + |y_2 - M| + |y_3 - M| + |y_4 - M|.$$

$$\boxed{SSR(M) = (y_1 - M)^2 + (y_2 - M)^2 + (y_3 - M)^2 + (y_4 - M)^2}$$

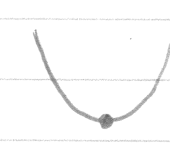
Sum of squares of Residuals

$$SSR'(M) = -2(y_1 - M) + \cdots + (-2)(y_4 - M) = 0$$

$$\leadsto M = \frac{y_1 + y_2 + y_3 + y_4}{4} \quad \text{(Check it's a min!)}$$

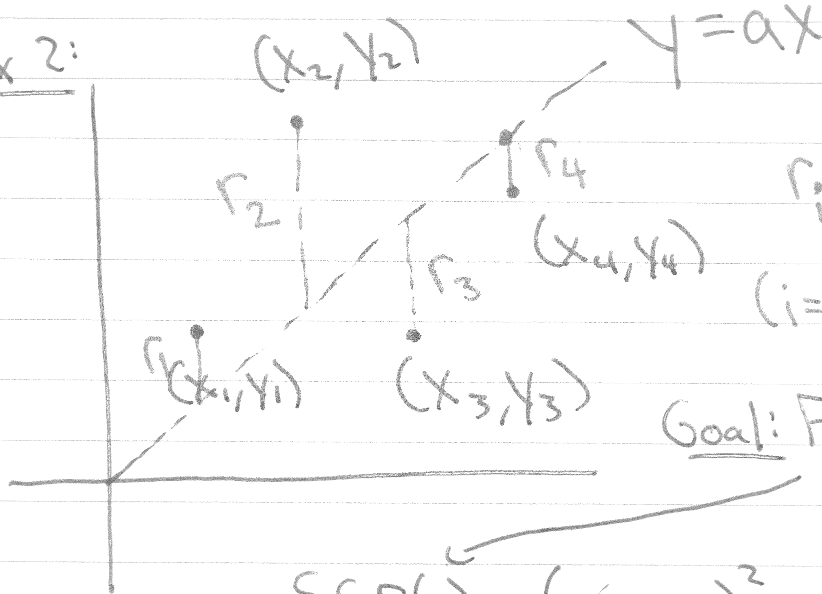
The average minimizes the sum of the squared residuals.

Check it's a minimum: $SSR''(M) = 2 + 2 + 2 + 2 = 8$



$\boxed{\text{Concave up.}}$

Ex 2:



$r_i = ax_i - y_i$

$(i=1,2,3,4)$

Goal: Find $\underline{\underline{a}}$ which minimizes

$$SSR(a) = (ax_1-y_1)^2 + (ax_2-y_2)^2 + (ax_3-y_3)^2 + (ax_4-y_4)^2$$

$$SSR'(a) = 2(ax_1-y_1)x_1 + 2(ax_2-y_2)x_2 + \cdots = 0$$

$$\cancel{2}a(x_1^2+x_2^2+x_3^2+x_4^2) - \cancel{2}(y_1x_1+y_2x_2+y_3x_3+y_4x_4) = 0$$

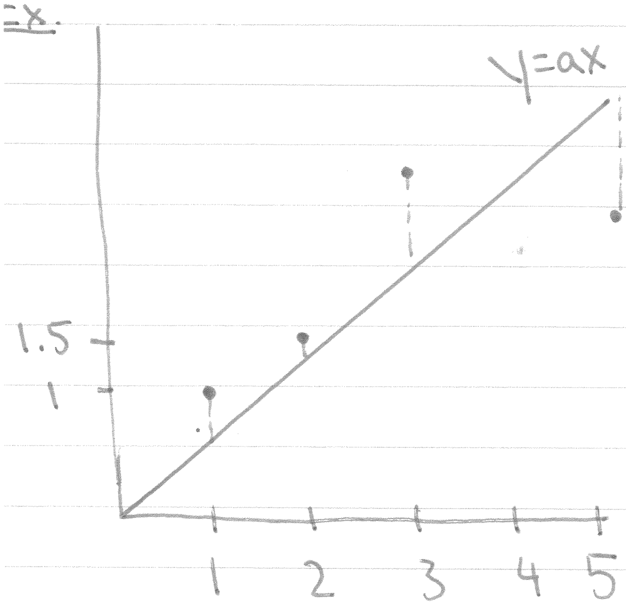
$$\boxed{a = \frac{x_1y_1+x_2y_2+x_3y_3+x_4y_4}{x_1^2+x_2^2+x_3^2+x_4^2}}$$

Intuition: $\frac{y_1}{x_1} = \frac{x_1y_1}{x_1^2}$ goes thru 1st line

$\frac{y_2}{x_2} = \frac{x_2y_2}{x_2^2}$ 2nd

$\vdots$

Ex.



(1,1), (2,1.5), (3,3), (5,4.5)

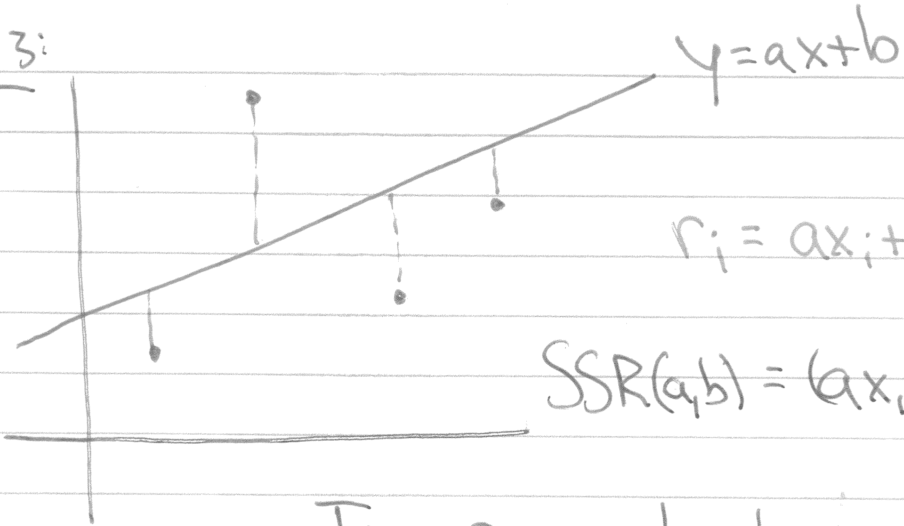
$$a = \frac{x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4}{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

$$= \frac{1\cdot 1 + 2\cdot 1.5 + 3\cdot 3 + 5\cdot 4.5}{1^2 + 2^2 + 3^2 + 5^2}$$

$$= \frac{1 + 3 + 9 + 22.5}{1 + 4 + 9 + 25} = \frac{35.5}{39} = \boxed{\frac{71}{78}}$$

The line $y = \frac{71}{78}x$ is the "best fit" line through the origin.

Ex 3:



$r_i = ax_i + b - y_i$

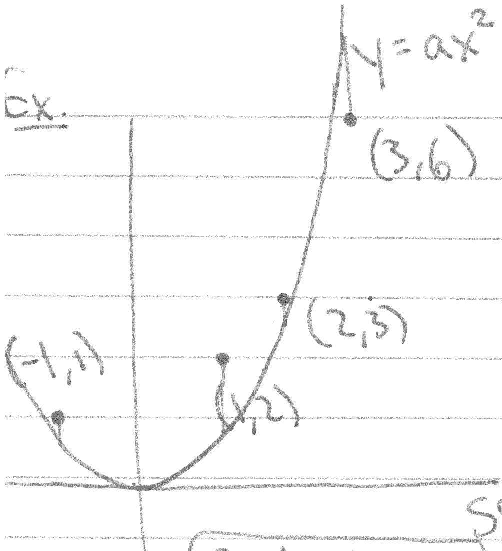
$SSR(a,b) = (ax_1 + b - y_1)^2 + \cdots$

Two parameters!

Multivariable calculus!

Fitting with a parabola $\underline{y = ax^2}$.

<u>Ex.</u>



ith residual $r_i = ax_i^2 - y_i$

$$SSR(a) = r_1^2 + r_2^2 + r_3^2 + r_4^2$$

$$= (ax_1^2 - y_1)^2 + (ax_2^2 - y_2)^2 + (ax_3^2 - y_3)^2 + (ax_4^2 - y_4)^2$$

$\boxed{\text{Critical point:}}$

$$SSR'(a) = 2x_1^2(ax_1^2 - y_1) + 2x_2^2(ax_2^2 - y_2) + 2x_3^2(ax_3^2 - y_3) + 2x_4^2(ax_4 - y_4) = 0$$

$$\Rightarrow ax_1^4 - x_1^2 y_1 + ax_2^4 - x_2^2 y_2 + ax_3^4 - x_3^2 y_3 + ax_4^4 - x_4^2 y_4 = 0$$

$$\Rightarrow a(x_1^4 + x_2^4 + x_3^4 + x_4^4) = x_1^2 y_1 + x_2^2 y_2 + x_3^2 y_3 + x_4^2 y_4$$

$$\Rightarrow \boxed{a = \frac{x_1^2 y_1 + x_2^2 y_2 + x_3^2 y_3 + x_4^2 y_4}{x_1^4 + x_2^4 + x_3^4 + x_4^4}}$$

This value of a gives the "best fit parabola $y = ax^2$"

<u>For (-1,1), (1,2), (2,3), (3,6),</u>

$$a = \frac{(-1)^2 \cdot 1 + 1^2 \cdot 2 + 2^2 \cdot 3 + 3^2 \cdot 6}{(-1)^4 + 1^4 + 2^4 + 3^4}.$$

$$= \frac{1 + 2 + 12 + 54}{1 + 1 + 16 + 81} = \frac{69}{99} = \boxed{\frac{23}{33}}$$

$$\rightsquigarrow y = \frac{23}{33}x^2$$